The Exponential map on the Cayley-Dickson algebras

Guillermo Moreno
Departamento de Matematicas.
CINVESTAV del I.P.N.
Mexico D.F. MEXICO
gmoreno@math.cinvestav.mx

Abstract: We study the Exponential map for $\mathbb{A}_n = \mathbb{R}^{2^n}$, the Cayley-Dickson algebras for $n \geq 1$, which generalize the complex exponential map to Quaternions, Octonions and so forth. As an application, we show that the selfmap of the unit sphere in $\mathbb{A}_n$, $S(\mathbb{A}_n) = S^{2^n-1}$, given by taking k-powers has topological degree k for k an integer number, from this we derive a suitable "Fundamental Theorem of Algebra for $\mathbb{A}_n$."

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Introduction. The Cayley-Dickson algebras $\mathbb{A}_n = \mathbb{R}^{2^n}$, for $n \geq 0$ are given by the doubling process of Dickson [1].

For a, b, x and y in $\mathbb{A}_n$ and $\mathbb{A}_{n+1} = \mathbb{A}_n \times \mathbb{A}_n$.

$$(a, b)(x, y) = (ax - \overline{y}b, ya + b\overline{x})$$

and for x_1, x_2 in $\mathbb{A}_{n-1}$, $\quad \overline{x} = (\overline{x}_1, -x_2)$ when $x = (x_1, x_2)$.

Thus, if $\overline{x} = x$ in $\mathbb{A}_0 = \mathbb{R}$ then $\mathbb{A}_1 = \mathbb{C}$ the Complex numbers, $\mathbb{A}_2 = \mathbb{H}$ the Quaternionic numbers; $\mathbb{A}_3 = \mathbb{O}$ the Octonions numbers and so forth.

As is well known (see [7] and [8]) $\mathbb{A}_n$ is commutative only for $n = 0$ and $n = 1$; $\mathbb{A}_n$ is associative only for $n = 0, 1, 2$; $\mathbb{A}_n$ is *alternative* (i.e., $x^2 y = x(xy)$ and $xy^2 = x(xy)$) for all x and y in $\mathbb{A}_n$) only if $n = 0, 1, 2, 3$.

Also $\mathbb{A}_n$ is *flexible* (i.e., $x(yx) = (xy)x$ for all x and y in $\mathbb{A}_n$) and *power−associative* (i.e. $x^m x^k = x^{m+k}$ for all x and y in $\mathbb{A}_n$ and m and k natural numbers) for all $n \geq 0$.

By the classical theorem of Hurwitz: $\mathbb{A}_n$ is *normed* (i.e., $||xy|| = ||x|| ||y||$ for all x and y in $\mathbb{A}_n$) if and only if $n = 0, 1, 2, 3$ where $|| - ||$ denotes the standard norm in $\mathbb{R}^{2^n}$.

Now $\{e_0 = 1, e_1, e_2, \ldots, e_{2^n - 1}\}$ denotes the canonical basis in $\mathbb{R}^{2^n}$

$$\mathbb{A}_n = \mathbb{R}e_0 \oplus \mathrm{Im}(\mathbb{A}_n)$$

where $\mathbb{R}e_0 = \mathrm{Span}\{e_0\}$ are the real elements in $\mathbb{A}_n$ and

$$\mathrm{Im}(\mathbb{A}_n) := \mathrm{Span}\{e_1, e_2, \ldots, e_{2^n - 1}\}$$

are the pure imaginary elements in $\mathbb{A}_n$. Thus,for every x in $\mathbb{A}_n$ we have a canonical splitting of x into real and imaginary parts: $x = re_0 + a$ where $r \in \mathbb{R}$ and $a \in \mathrm{Im}(\mathbb{A}_n)$.

Now for all x in $\mathbb{A}_n$ $||x||^2 = x\overline{x} = \overline{x}x$ and $2\langle x, y\rangle = x\overline{y} + y\overline{x}$ where $\langle -, - \rangle$ denotes the standard inner product in $\mathbb{R}^{2^n}$. Thus for $a \in \mathrm{Im}(\mathbb{A}_n)$, $\quad \overline{a} = -a$ and $a^2 = -||a||^2$ and

$$\langle xy, z\rangle = \langle y, \overline{x}z\rangle \quad \text{and} \langle x, yz\rangle = \langle x\overline{z}, y\rangle$$

for all x, y and z in $\mathbb{A}_n$ (see [5]).

The main subject of this paper is the Exponential map

$$\exp(x) = e_0 + x + \frac{x^2}{2!} + \ldots \frac{x^m}{m!} + \ldots = \sum_{m=0}^{\infty} \frac{x^m}{m!} \quad \text{for} \quad x \in \mathbb{A}_n.$$

In § 1, we prove that $\exp(x)$ is well defined, that is, converges for all $n \geq 0$ and that some properties of the complex exponential map are also valid for $n > 1$.

In §2 we show that the exponential map is surjective, and that restricted to $\mathrm{Im}(\mathbb{A}_n)$ it maps onto $S(\mathbb{A}_n) = S^{2^n-1}$, the unit sphere in $\mathbb{A}_n$; also we prove that the k-power map is well defined for $k \in \mathbb{Z}$ and has topological degree k.

In § 3 we use the results of the previous sections to prove the Fundamental Theorem of Algebra (F.T.of A.) for $\mathbb{A}_n \quad n \geq 2$, which generalizes the F.T. of A. for Quaternions of Eilenberg-Niven [2] that goes back to 1949.

First, we present an *Algebraic* generalization of the F.T. of A. which only considers polynomials where the variable and the coefficients have linearly dependent imaginary part. This version of the F.T. of A. is a straigthforward generalization of the classical F.T. of A. for $\mathbb{C}$.

Secondly, we present a *Topological* generalization of the F.T. of A. for "polynomials of degree k" which are continuous functions inside the homotopy class of the k-power map on $\mathbb{A}_n \cup \{\infty\} = S^{2^n}$ for each $k > 0$.

We will show, as well, that the topological F.T. of A. generalizes the Algebraic F. T. of A.

I. Basic Properties.

Throughtout this paper, we use extensively that $\mathbb{R}e_0$ is the Center of $\mathbb{A}_n$ for all $n \geq 1$ and that for x, y and z in $\mathbb{A}_n \quad (xy)z = x(yz)$ if at least one of them is a real element.

Definition: For $x \in \mathbb{A}_n, \exp(x) = e_0 + x + \frac{x^2}{2!} + \cdots = \sum_{m=0}^{\infty} \frac{x^m}{m!}$

where e_0 is the unit element in $\mathbb{A}_n$.

Clearly for $x = re_0$ for $r \in \mathbb{R}$, $\exp(x) = e^r e_0$ where e^r is the usual real exponent map on $\mathbb{A}_0$. In particular if 0 in $\mathbb{A}_n$ is the null element

$$\exp(0) = e^0 e_0 = 1 \cdot e_0 = e_0$$

.

Lemma 1.1 If a is a non-zero element in $\mathrm{Im}(\mathbb{A}_n)$ then

$$\exp(a) = \cos(||a||)e_0 + \sin(||a||)\frac{a}{||a||}.$$

Proof: Since $a \in \mathrm{Im}(\mathbb{A}_n)$ then $a^2 = -||a||^2$ so $a^{2k} = (-1)^k ||a||^{2k}$ and $a^{2k+1} = a^{2k}a = (-1)^k ||a||^{2k} a$ for $k \geq 0$. Therefore

$$\begin{aligned}
\exp(a) &= \sum_{m=0}^{\infty} \frac{a^m}{m!} = \sum_{k=0}^{\infty} \frac{(-1)^k ||a||^{2k}}{(2k)!} e_0 + \sum_{k=0}^{\infty} \frac{(-1)||a||^{2k}}{(2k+1)!} a \\
&= \cos(||a||)e_0 + \frac{1}{||a||} \sum_{k=0}^{\infty} \frac{(-1)^k ||a||^{2k+1}}{(2k+1)!} a \\
&= \cos(||a||)e_0 + \sin(||a||) \frac{a}{||a||}
\end{aligned}$$

Q.E.D.

Corollary 1.2 For non-zero s in $\mathbb{R}$ and a non-zero in $\mathrm{Im}(\mathbb{A}_n)$

$$||\exp(a)|| = 1 \quad \text{and} \quad \exp(sa) = \cos(s||a||) + \sin(s||a||)\frac{a}{||a||}.$$

Proof: $||\exp(a)||^2 = \cos^2(||a||) + \sin^2(a)\frac{||a||}{||a||} = 1$.

Since $\frac{s}{||s||}$ is equal to 1 for $s > 0$ and equal to -1 for $s < 0$ and cosine and sine are even and odd function respectively then

$$\begin{aligned}
\exp(sa) &= \cos(||sa||)e_0 + \sin(||sa||)\frac{sa}{||sa||} \\
&= \cos(|s|||a||)e_0 + \sin(|s|||a||)\frac{sa}{|s|||a||} \\
&= \cos(s||a||)e_0 + \sin(s||a||)\frac{a}{||a||}.
\end{aligned}$$

Q.E.D.

Theorem 1.3 For x in $\mathbb{A}_n$ and $n \geq 1$, the series $\exp(x)$ converges.

If $x = re_0 + a$ for $r \in \mathbb{R}$ and a in $\mathrm{Im}(\mathbb{A}_n)$ then

$$\exp(x) = e^r(\exp(a))$$

where e^r is the real exponent map.

Proof: A direct calculation shows that

$$e^r(\exp(a)) = (\sum_{m=0}^{\infty} \frac{r^m}{m!})(\sum_{m=0}^{\infty} \frac{a^m}{m!})$$

$$
\begin{aligned}
&= (e_0 + re_0 + \frac{r^2 e_0}{2!} + \cdots)(e_0 + a + \frac{a^2}{2!} + \cdots) \\
&= e_0 + re_0 + a + \frac{r^2 e_0}{2!} + ra + \frac{a^2}{2!} + \cdots \\
&= e_0 + (re_0 + a) + \frac{(re_0 + a)^2}{2!} + \cdots \\
&= \exp(x)
\end{aligned}
$$

Q.E.D.

Corollary 1.4. For x in $\mathbb{A}_n$, $||\exp(x)|| = e^r$, where r is the real part of x.

Proof:

$$||\exp(x)|| = ||e^r \exp(a)|| = |e^r| ||\exp(a)|| = |e^r| = e^r.$$

by Theorem 1.3 and Corollary 1.2.

Q.E.D.

Example: The known complex identities $e^{\frac{i\pi}{2}} = i$ and $e^{i\pi} = -1$ correspond in $\mathbb{A}_n$ for $n \geq 1$ to $\exp(\frac{a\pi}{2}) = a$ and $\exp(\pi a) = -e_0$ respectively for every a in $\mathrm{Im}(\mathbb{A}_n)$ such that $||a|| = 1$.

Now we show that,the identity $e^{z+w} = e^z \cdot e^w$ in $\mathbb{C}$ can be generalized to $\mathbb{A}_n$ for $n \geq 1$, to certain extent.

Definition: Two elements in $\mathbb{A}_n$ for $n \geq 1$ are <u>Complex dependent</u> or <u>$\mathbb{C}$-dependent</u> if their respective pure imaginary parts are linearly dependent.

Notice that, every element in $\mathbb{A}_n$ is $\mathbb{C}$-dependent with any real element and that, for $n = 1$ every two elements are $\mathbb{C}$-dependent, because $\mathrm{Im}(\mathbb{A}_1) = \mathbb{R}e_1$.

Also notice that for every x in $\mathbb{A}_n$, $\exp(x)$ and x are $\mathbb{C}$-dependent.

Lemma 1.5 Let x and y be in $\mathbb{A}_n$ for $n \geq 1$.

(i) If x and y are $\mathbb{C}$-dependent then $xy = yx$.

(ii) For $n = 2$ and $n = 3$, we have that, x and y are $\mathbb{C}$-dependent if and only if $xy = yx$.

Proof: First of all, we observe that two elements in $\mathbb{A}_n$ commute if and only if their respective imaginary parts commute.

Let $x = re_0 + a$ and $y = se_0 + b$ in $\mathbb{R}e_0 \oplus \mathrm{Im}(\mathbb{A}_n) = \mathbb{A}_n$. So

$$\begin{aligned} xy &= (re_0 + a)(se_0 + b) = (rse_0 + rb + sa + ab) \\ yx &= (se_0 + b)(re_0 + a) = (sre_0 + sa + rb + ba) \end{aligned}$$

Then $xy = yx$ if and only if $ab = ba$.

Suppose that $b = ta$ with t in $\mathbb{R}$ then $ab = a(ta) = (ta)a = ba$ and we are done with (1).

Now notice that if $ab = ba$ for a and b in $\mathrm{Im}(\mathbb{A}_n)$ then (ab) is real, because $\overline{ab} = \overline{b}\overline{a} = (-b)(-a) = ba = ab$. Moreover, since $2\langle a, b\rangle = a\overline{b} + b\overline{a} = -ab - ba = -2(ab)$ then $\langle a, b\rangle e_0 = -ab$.

To prove (ii) recall that for $n = 2$ and $n = 3$ we have that $a(ab) = a^2 b$ so $a(ab) = -a(\langle a, b\rangle e_0)$ implies $a^2 b = -||a||^2 b = -\langle a, b\rangle a$ and a and b are linearly dependent.

Q.E.D.

Remarks: 1.-Notice that the presence of <u>zero divisors</u> in $\mathbb{A}_n$ for $n \geq 4$, makes possible to have non-zero elements a and b in $\mathrm{Im}(\mathbb{A}_n)$ with $ab = ba = 0$ and a orthogonal to b (see [5]).

2.- A further study shows that for a in $\mathrm{Im}(\mathbb{A}_n)$ and $n \geq 4$, the <u>Centralizer</u> of a, defined by $C_a := \{b \in \mathrm{Im}(\mathbb{A}_n) | ab = ba\}$ is given by

$$C_a = \mathbb{R}a \oplus KerL_a.$$

where $\underline{KerL_a}$ is the right annihilator of a.

3.- A characterization of $\mathbb{C}$-dependence is given by:For x and y in $\mathbb{A}_n$.

x and y are $\mathbb{C}$-dependent if and only if $x(zy) = (xz)y$ for all z in $\mathbb{A}_n$. (See [3], [4] and [6]). (We will not use this).

Lemma 1.6 Let a and b be in $\mathrm{Im}(\mathbb{A}_n)$ for $n \geq 2$. If a and b are linearly dependent then

$$\exp(a + b) = \exp(a)\exp(b).$$

Proof: Notice that if either a or b is null then the assertion is trivial.

Suppose that neither a nor b are null then there is non-zero real number s such that $b = sa$ and

$$\begin{aligned} \exp(b) &= \cos(||sa||)e_0 + \sin(||sa||)\frac{sa}{||sa||} \quad \text{(Lemma1.1).} \\ &= \cos(s||a||)e_0 + \sin(s||a||)\frac{a}{||a||} \text{(Corollary1.2)} \end{aligned}$$

Now by the standard trigonometric identities for addition of angles for cosine and sine we have that

$$\begin{aligned}
\exp(a)\exp(b) &= (\cos(||a||)e_0 + \sin(||a||)\frac{a}{||a||})(\cos(s||a||) + \sin(s||a||)\frac{a}{||a||}) \\
&= [\cos(||a||)\cos(s||a||) - \sin(||a||)\sin(s||a||)]e_0 \\
&+ [\cos(||a||)\sin(s||a||) + \sin(||a||)\cos(s||a||)]\frac{a}{||a||} \\
&= [\cos(||a|| + s||a||))e_0 + \sin(||a|| + s||a||))\frac{a}{||a||} \\
&= (\cos(||a + sa||))e_0 + (\sin(||a + sa||))\frac{a + sa}{||a + sa||} \\
&= \exp(a + b)
\end{aligned}$$

because of $a^2 = -||a||^2 e_0$ and Corollary 1.2.

Q.E.D.

Theorem 1.7 If x and y are $\mathbb{C}$-dependent in $\mathbb{A}_n$ and $n \geq 1$ then

$$\exp(x + y) = \exp(x)\exp(y)$$

Proof: Suppose that $x = re_0 + a$ and $y = se_0 + b$ in $\mathbb{R}e_0 \oplus \mathrm{Im}(\mathbb{A}_n)$ so $x + y = (r + s)e_0 + (a + b)$.

$$\begin{aligned}
\exp(x + y) &= e^{r+s}(\exp(a + b)) = e^{r+s}(\exp(a)\exp(b)) \\
&= (e^r\exp(a))(e^s\exp(b)) \\
&= \exp(x)\exp(y).
\end{aligned}$$

Q.E.D.

Remark: Theorem 1.7 is the best possible in the following sense:

For $n = 1$. Theorem 1.7 correspond to the exponential law.

For $n \geq 2$. Assume that $\exp(x + y) = \exp(x)\exp(y)$ for some x and y in $\mathbb{A}_n$ then we are forced to have, at least, the following two conditions: $xy = yx$ and $x(xy) = x^2y$.

Now if a and b are the pure parts of x and y respectively, then $xy = yx$ if and only if $ab = ba$ and $x^2y = x(xy)$ if and only if $a^2b = a(ab)$.(See [6])

If $ab = ba$ then $ab = -\langle a, b\rangle e_0$ so $-||a||^2 b = a^2 b = a(ab) = -\langle a, b\rangle e_0$ and a and b are linearly dependent. (Notice that $ab \neq 0$ because $a(ab) = a^2 b \neq 0$).

Proposition 1.8. For x in $\mathbb{A}_n$ and $n \geq 1$.
(i) $\exp(\overline{x}) = \overline{\exp(x)}$.
(ii) $\exp(-x) = \exp(x)^{-1}$

Proof: Suppose that $x = re_0 + a$ in $\mathbb{R}e_0 \oplus \mathrm{Im}(\mathbb{A}_n)$ then $\overline{x} = re_0 - a$ and

$$\begin{aligned} \exp(\overline{x}) &= \exp(re_0 - a) = e^r(\exp(-a)) = e^r(\cos(||a||)e_0 + \sin(||a||)(-\frac{a}{||a||}) \\ &= e^r((\cos(||a||)e_0 - \sin(||a||)\frac{a}{||a||}) \\ &= \overline{\exp(x)}. \end{aligned}$$

so we are done with (i).

To prove (ii) recall for non-zero y in $\mathbb{A}_n$, by definiton, $y^{-1} = ||y||^{-2}\overline{y}$ so

$$\begin{aligned} \exp(x)^{-1} &= ||\exp(x)||^{-2r}\exp(\overline{x}) \quad \text{by}(1) \\ &= e^{-2r}(e^r(\exp(-a))) = e^{-r}\exp(-a) \\ &= \exp(-x) \end{aligned}$$

Corollary 1.9 For x in $\mathbb{A}_n$ and k in $\mathbb{Z}$.

$$\exp(kx) = (\exp(x))^k \quad \text{(De Moivre's Formula).}$$

Proof: For $k = 0$. The assertion is trivial.

For $k > 0$. The proof is straight-forward using the addition of angles identities, for sine and cosine respectively and induction on k.

For $k < 0$. The proof follows from case $k > 0$ and Proposition 1.8 (ii).

Q.E.D.

II. The exponential map the and k-power map.

$S(\mathbb{A}_n) = S^{2^n-1}$ denotes the unit sphere in $\mathbb{A}_n = \mathbb{R}^{2^n}$.

Theorem 2.1 The exponential map $\exp:\mathbb{A}_n \to \mathbb{A}_n\backslash\{0\}$ and its restriction $\exp:\mathrm{Im}(\mathbb{A}_n) \to S(\mathbb{A}_n)$ are onto maps for all $n \geq 1$.

Proof: By Corollary 1.2 we know that exp $(\mathrm{Im}\,(\mathbb{A}_n)) \subset S(\mathbb{A}_n)$.

Suppose that $y = se_0 + b$ for b in $\mathrm{Im}(\mathbb{A}_n)$ and s in $\mathbb{R}$ with $||y||^2 = s^2 + ||b||^2 = 1$.

If $b = 0$ then $s^2 = 1$ and $s = \mp 1$ and $y = \mp e_0$ but $\exp(0) = e_0$ and $\exp(\pi c) = -e_0$ for all $c \in S(\mathbb{A}_n)$.

Suppose that $b \neq 0$ in Im $(\mathbb{A}_n)$ then there is a real number θ such that $0 < \theta < \pi$ with $s = \cos(\theta)$ and $||b|| = \sin(\theta)$.

Let us define a as the non-zero element in Im $(\mathbb{A}_n)$ of norm θ and linearly dependent to b, this means,

$$||a|| = \theta \quad \text{and} \quad ||b||(\frac{a}{||a||}) = b.$$

Therefore

$$\begin{aligned}
\exp(a) &= \cos(||a||)e_0 + \sin(||a||)\frac{a}{||a||} \\
&= \cos(\theta)e_0 + \sin(\theta)\frac{b}{||b||} \\
&= se_0 + b \\
&= y.
\end{aligned}$$

Therefore exp:$\mathrm{Im}(\mathbb{A}_n) \to S(\mathbb{A}_n)$ is onto.

Now suppose that $y \neq 0$ in $\mathbb{A}_n$ then $||y||^{-1}y$ is in $S(\mathbb{A}_n)$ so there is a in $\mathrm{Im}(\mathbb{A}_n)$ such that $\exp(a) = ||y||^{-1}y$ then

$$||y||\exp(a) = y.$$

But the real exponent map $e : \mathbb{R} \to \mathbb{R}^+$ is onto, so there is r in $\mathbb{R}^+$ such that $||y|| = e^r$ and if $x = re_0 + a$ we have that

$$\exp(x) = e^r\exp(a) = y$$

Q.E.D.

Now we study the k-power map $x \mapsto x^k$ for k in $\mathbb{Z}$.

Lemma 2.2 For $k > 0$ and x in $\mathbb{A}_n$ we have

(1) $\overline{(x^k)} = (\overline{x})^k$

(2) $||x||^k = ||x^k||$
(3) If $x \neq 0$ then $(x^{-1})^k = (x^k)^{-1}$

Proof: Notice that for $k = 1$ (1), (2) and (3) are trivial. So we assume that $k \geq 2$.

(1) We proceed by induction on k.

Recall that for x and y in $\mathbb{A}_n$, $\overline{xy} = \overline{y}\overline{x}$ so $(\overline{x})^2 = (\overline{x})(\overline{x}) = \overline{xx} = \overline{(x^2)}$. Suppose now that $(\overline{x})^k = \overline{(x^k)}$. Then $(\overline{x})^{k+1} = (\overline{x})^k(\overline{x}) = \overline{(x^k)}\overline{x} = \overline{xx^k} = \overline{x^{k+1}}$ and (1) is done.

(2) Notice that for x and y in $\mathbb{A}_n$ if $\overline{x}(xy) = (\overline{x}x)y = ||x||^2 y$ then

$$||xy|| = ||x||||y||$$

because

$$||xy||^2 = \langle xy, xy\rangle = \langle y, \overline{x}(xy)\rangle = \langle y, ||x||^2 y\rangle = ||x||^2\langle y, y\rangle = ||x||^2||y||^2$$

We also notice that $\overline{x}(xy) = (\overline{x}x)y$ if and only if $x(xy) = x^2y$, because $(\overline{x} + x)$ is real and hence associates with any other two elements in $\mathbb{A}_n$ so, in particular

$$\overline{x}(xy) + x(xy) = (\overline{x} + x)(xy) = ((\overline{x} + x)x)y = (\overline{x}x)y + x^2y = ||x||^2y + x^2y$$

therefore $\overline{x}(xy) - ||x||^2y = x^2y - x(xy)$.

Making $y = x^{k-1}$ for $k \geq 2$ and recalling that $\mathbb{A}_n$ is power associative, we have that $||x^{k+1}|| = ||x(xx^{k-1})|| = ||(x^2)||||(x^{k-1})||$ for $k \geq 2$. By an obvious induction we are done with (2).

(3) Recall that for $y \neq 0$ in $\mathbb{A}_n$ $\quad y^{-1} = ||y||^{-2}\overline{y}$.Using (1) and (2) we have that for $x \neq 0$

$$(x^{-1})^k = (||x||^{-2}\overline{x})^k = ||x||^{-2k}(\overline{x})^k = (||x||^k)^{-2}\overline{x}^k) = ||x^k||^{-2}(\overline{x}^k) = (x^k)^{-1}$$

Q.E.D.

Definition: For nonzero x, $x^{-k} := (x^{-1})^k = (x^k)^{-1}$.

For $k > 0 \quad \rho_k := \mathbb{A}_n \setminus \{0\} \to \mathbb{A}_n$ is $\rho_k(x) = x^k$.
For $k = 0 \quad \rho_0 : \mathbb{A}_n \setminus \{0\} \to \mathbb{A}_n$ is $\rho_0(x) = e_0$.
For $k < 0 \quad \rho_k : \mathbb{A}_n \setminus \{0\} \to \mathbb{A}_n$ is $\rho_k(x) = \rho_{-k}(x^{-1})$.

ρ_k is, by definition, <u>the k–power map</u> for k in $\mathbb{Z}$; and clearly $\rho_k \circ \rho_\ell = \rho_{k+\ell}$ for k and ℓ in $\mathbb{Z}$.

Theorem 2.3 For $k \in \mathbb{Z}$ and $\rho_k : S(\mathbb{A}_n) \to S(\mathbb{A}_n)$ has topological degree k.

Proof: By Lemma 2.2 (2) we have that $\rho_k(S(\mathbb{A}_n)) \subset S(\mathbb{A}_n)$.

Now define $\sigma_k : \mathbb{A}_n \to \mathbb{A}_n$ as $\sigma_k(x) = kx$ for $k \in \mathbb{Z}$. Now by Corollary 1.9 and Lemma 2.2 (2) and (3) $\exp(\sigma_k(x)) = \rho_k(\exp(x))$.

Therefore Im $(\mathbb{A}_n)$ can be seen, as the tangent space of $S(\mathbb{A}_n)$ at $x = e_0$, so degree of ρ_k is k.

Q.E.D.

Now we study the k–power map $\rho_k : S(\mathbb{A}_{n+1}) \to S(\mathbb{A}_{n+1}) = S^{2^{n+1}-1}$ restrcited to a subsphere of dimension 2^n for $n \geq 1$.

Consider the vector subspace of $\mathbb{A}_{n+1} = \mathbb{A}_n \times \mathbb{A}_n$ where the second coordinate is real in $\mathbb{A}_n$ that is

$$\mathbb{A}_n \times \mathbb{A}_0 := \{(x, y)r \in \mathbb{A}_n \times \mathbb{A}_n | y = re_0 \text{ for } \in \mathbb{R}\}.$$

Clearly $\mathbb{A}_n \times \mathbb{A}_0$ is a vector subspace of $\mathbb{A}_{n+1}$, which is closed under conjugation and inverses; i.e., if $(x, re_0) \in \mathbb{A}_n \times \mathbb{A}_0$ then $\overline{(x, re_0)} = (\overline{x}, -re_o) \in \mathbb{A}_n \times \mathbb{A}_0$ and for $(x, re_0) \neq (0, 0)$ then $(x, re_0)^{-1} \in \mathbb{A}_n \times \mathbb{A}_0$.

Lemma 2.4 The vector subspace $\mathbb{A}_n \times \mathbb{A}_0$ of $\mathbb{A}_{n+1}$ is closed under k-powers for k in $\mathbb{Z}$.

Proof: Clearly the cases $k = 0$ and $k = 1$ are obvious. Based on the above observation,the case $k < 0$ follows from the case $k > 0$.

First we check that $\mathbb{A}_n \times \mathbb{A}_0$ is closed under squaring operation

$$(x, re_0)^2 = (x, re_0)(x, re_0) = (x^2 - r^2 e_0, rx + r\overline{x}) = (x^2 - r^2 e_0, r(x + \overline{x}))$$

but $(x + \overline{x})$ is real so $(x, re_0)^2 \in \mathbb{A}_n \times \mathbb{A}_0$.

Now we proced by induction on k for $k \geq 2$.

Suppose that $\alpha \in (\mathbb{A}_n \times \mathbb{A}_0)$ and that $\alpha^k \in (\mathbb{A}_n \times \mathbb{A}_0)$.

We want to prove that $\alpha^{k+1} \in (\mathbb{A}_n \times \mathbb{A}_0)$.

Since $(\mathbb{A}_n \times \mathbb{A}_0)$ is vector subspace then $(\alpha^k + \alpha) \in (\mathbb{A}_n \times \mathbb{A}_0)$ and because $(\mathbb{A}_n \times \mathbb{A}_0)$ is closed under the squaring operation we have that $(\alpha^k + \alpha)^2, (\alpha^k)^2 = \alpha^{2k}$ and α^2 are in $\mathbb{A}_n \times \mathbb{A}_0$.

But, we can associate powers in $\mathbb{A}_{n+1}$ so

$$(\alpha^k + \alpha)^2 = \alpha^{2k} + 2\alpha^{k+1} + \alpha^2.$$

Therefore $\alpha^{k+1} \in \mathbb{A}_n \times \mathbb{A}_0$

Q.E.D.

Remark: We can prove a more general assertion than the one in lemma 2.4.

Let V be a vector subspace of $\mathbb{A}_n$ and define the following vector subspace of $\mathbb{A}_{n+1}$.

$$\mathbb{A}_n \times V = \{(x, v) \in \mathbb{A}_n \times \mathbb{A}_n | v \in V\}$$

Clearly these vector subspace is closed under conjugation and inverses and

$$(x, v)^2 = (x^2 - ||v||^2 e_0, v(\overline{x} + x))$$

But $(\overline{x} + x)$ is real so $\mathbb{A}_n \times V$ is closed under squares by similar argument as in Lemma 2.4. we have that $\mathbb{A}_n \times V$ is closed under k powers for k in $\mathbb{Z}$.

Corollary 2.5 The k–power map $\rho_k : S(\mathbb{A}_{n+1}) \to S(\mathbb{A}_{n+1})$ restricted to $S(\mathbb{A}_n \times \mathbb{A}_0) = S^{2^n}$ has also topological degree k for $k \in \mathbb{Z}$.

Proof: Recall that $S(\mathbb{A}_{n+1}) = \{(x, y) \in \mathbb{A}_n \times \mathbb{A}_n | ||x||^2 + ||y||^2 = 1\}$ thus $S(\mathbb{A}_n \times \mathbb{A}_0) = \{(x, re_0) \in \mathbb{A}_n \times \mathbb{A}_0 | ||x||^2 + r^2 = 1\}$.

By Lemma 2.4 and Lemma 2.2 (2) $\rho_k(S(\mathbb{A}_n \times \mathbb{A}_0)) \subset S(\mathbb{A}_n \times \mathbb{A}_0)$ and by Theorem 2.3 $\rho_k : S(\mathbb{A}_n \times \mathbb{A}_0) \to S(\mathbb{A}_n \times \mathbb{A}_0)$ has degree k.

Q.E.D.

Remark: Based in the previous remark we may prove that the k power map on $S(\mathbb{A}_{n+1})$ restricted to $S(\mathbb{A}_n \times V)$ has also degree k.

Therefore any continuos map from S^m to itself is homotopic to a k power map for all $m \geq 1$,because the homotopy class of any continuos map is determined by degree (Hopf theorem) and if m is between 2^n and 2^{n+1} we may choose a vector subspace V of $\mathbb{A}_n$ such that $S^m = S(\mathbb{A}_n \times V)$ and the restriction of $\rho_k : S(\mathbb{A}_{n+1}) \to S(\mathbb{A}_{n+1})$ to $S(\mathbb{A}_n \times V)$ is homotopic to the

original map.

III. Fundamental theorem of algebra for $\mathbb{A}_n$ $n \geq 1$.

Definition: For x non–zero in $\mathbb{A}_n, x = re_0 + a$ in $\mathbb{R}e_0 \oplus \mathrm{Im}(\mathbb{A}_n)$ and $k > 0$ <u>a k-root of x</u> is

$$\sqrt[k]{x} = ||x||^{1/k} \exp(\frac{1}{k}a) = e^{r/k} \exp(\frac{1}{k}a).$$

By Theorem 2.1 every non–zero element in $\mathbb{A}_n$ has (at least) one k–root for $k > 0$.

Example: $x^2 + e_0 = 0$ has infinitely many solutions: every element in in the unit sphere, $S(\mathrm{Im}(\mathbb{A}_n))$ is a solution.

Example: If a is a non–zero pure element in $\mathbb{A}_n$ then $x^k - a = 0$ has exactly k solutions, namely, the set of k–roots of a.

Now we want to extend the fundamental theorem of algebra for $\mathbb{A}_1 = \mathbb{C}$ to $\mathbb{A}_n$ for $n > 1$.

One direct generalization, can be done, on the polynomials which depend only on one imaginary unit in $\mathbb{A}_n$. That is, we look at the polynomials which are $\mathbb{C}$-dependent to a given a in Im $(\mathbb{A}_n)$ with $||a|| = 1$. So a plays the same role, for this polynomials, as $e_1 = i$ plays for complex polynomials and we have a Fundamental Theorem of Algebra in this situation.

Lemma 3.1 If x, y and z are non-zero $\mathbb{C}$-dependent elements in $\mathbb{A}_n$ then

(i) xy is $\mathbb{C}$- dependent with z.

(ii) x^k and y^ℓ are $\mathbb{C}$-dependent for $k > 0$ and $\ell > 0$

(iii) $(xy)z = x(yz)$.

Proof: If one of the three elements is real then the results (i), (ii) and (iii) are obvious.

Suppose that, the three elements x, y and z are non-real, that is, they have non–zero imaginary part. Also, is easy to see, that on the subset of $\mathbb{A}_n$ consisting of non-real elements, $\mathbb{C}$–dependence define an equivalence relation.

Write $x = re_0 + ta$ $y = se_0 + qa$ and $z = ue_0 + va$ where r, t, s, q, u and v are real numbers and $a \in \mathrm{Im}(\mathbb{A}_n)$ with $||a|| = 1$.

Now

i) $xy = (re_0 + ta)(se_0 + qa) = (rs - tq)e_0 + (rq + ts)a$ and (xy) is $\mathbb{C}$-dependent with z.

   To show (ii) we notice that x is $\mathbb{C}$-dependent with y^2,because

$$y^2 = (s^2 - q^2)e_0 + (s + q)a$$

.

   Next we proced by induction on k.

   Suppose that x is $\mathbb{C}$-dependent with x^k and we want to prove that x is $\mathbb{C}$-dependent with x^{k+1}.

   Now $(x^k + x)$, $(x^k + x)^2$, $(x^k)^2$ and x^2 are $\mathbb{C}$-dependent with x so $(x^k + x)^2 - (x^k)^2 - x^2 = 2x^{k+1}$ and x is $\mathbb{C}$-dependent with x^{k+1}.

   Since $\mathbb{C}$-dependence is an equivalence relation for non-real elements we are done with (ii).

   To prove (iii) recall that (see [5]) the associator $(x, y, z) := (xy)z - x(yz)$ is a tri-linear map that vanish if one of the entries is real so by flexibilty

   $(x, y, z) = (re_0 + ta, se_0 + qa, ue_0 + va) = tqv(a, a, a) = 0.$

Q.E.D.

Definition: A complex polynomial in $\mathbb{A}_n$ of degree k is a continuous function of the form

$$p(x) = \xi_0 + \xi_1 x + \xi_2 x^2 + \cdots + \xi_{k-1}x^{k-1} + x^k$$

where the coefficents ξ_i are $\mathbb{C}$–dependent among them and with x, and $i = 0, 1, \ldots, k - 1$.

Notice that every polynomial with real coeficients is a complex polynomial.

Theorem 3.2 Every complex polynomial has at least one root in $\mathbb{A}_n$.

Proof: This follows from the Fundamental Theorem of Algebra for $\mathbb{C}$.

Suppose that $x = re_0 + sa$ where $0 \neq s$ and r in $\mathbb{R}$ and $a \in \mathrm{Im}(\mathbb{A}_n)$ with $||a|| = 1$ so $p(x)$ and all the summands in $p(x)$ are in the complex subspace of $\mathbb{A}_n$ generated by $\{e_0, a\}$, because, Lemma 3.1 (i), (ii) and (iii).

Suppose that $x = re_0$, that is, $s = 0$, so by definition the polynomial is a real polynomial and it has at least one complex root and we may choose any $a \in S(\mathbb{A}_n)$ to inmerse the polynomial into $\mathrm{Span}\{e_0, a\}$.

Q.E.D.

Now we use what we know about the topology of the k–power map to extend the Fundamental theorem of algebra to a more general type of continuous functions than the complex polynomial in $\mathbb{A}_n$

Before that, we show, that some polynomials have no roots in $\mathbb{A}_n$.

Exmaple: For $n \geq 2$ and non–zero a in $\mathrm{Im}(\mathbb{A}_n)$

$$p(x) = ax - xa + e_0$$

has no roots in $\mathbb{A}_n$.Because every commutator of this form,$[a, x] = ax - xa$ has real part equal to zero.

Definition: A <u>generalized polynomial</u> of degree k on $\mathbb{A}_n$ with $k > 0$ is a continuous function $p : \mathbb{A}_n \setminus \{0\} \to \mathbb{A}_n$ of the form

$$p(x) = x^k(e_0 + g(x)).$$

where $g(x)$ is a nonconstant continuous function,defined for non-zero elements in $\mathbb{A}_n$, such that $||g(x)|| \to 0$ when $||x|| \to \infty$.

Proposition 3.3 Every complex polynomial in $\mathbb{A}_n$ for $n \geq 1$ is a generalized polynomial in $\mathbb{A}_n$.

Proof: Suppose that ξ and $x \neq 0$ are $\mathbb{C}$–dependet in $\mathbb{A}_n$ then by lemma 3.1 and lemma 1.5 (1) we have that

$$x^{-k}(\xi x^\ell) = (x^{-k}\xi)x^\ell = (\xi x^{-k})x^\ell = \xi(x^{-k}x^\ell) = \xi x^{-k+\ell}$$

for k in $\mathbb{Z}$ and $\ell \geq 0$.

Therefore if $p(x) = \xi_0 + \xi_1 x + \cdots + \xi_{k-1}x^{k-1} + x^k$ is a complex polynomial and $g(x) := x^{-k}(\xi_0 + \xi_1 x + \cdots + \xi_k x^{k-1}) = \xi_0 x^{-k} + \xi_1 x^{-k+1} + \cdots + \xi_k x^{-1}$ then $||g(x)|| \to 0$ when $||x|| \to \infty$ and $p(x) = x^k(e_0 + g(x))$ by Lemma 3.1 (iii).

Q.E.D.

Theorem 3.4 (Fundamental theorem of algebra for $\mathbb{A}_n$). Every generalized polynomial has at least one root in $\mathbb{A}_n$,for $n \geq 1$.

Proof: Given $p(x)$ a generalized polynomial in $\mathbb{A}_n$ define

$$\hat{p} : \mathbb{A}_n \cup \{\infty\} = S^{2^n} \to \mathbb{A}_n \cup \{\infty\} = S^{2^n}$$

with

$$\hat{p}(x) = \begin{cases} p(x) & \text{if} \quad ||x|| < \infty \\ \infty & \text{if} \quad x = \infty \end{cases}$$

where $\mathbb{A}_n \cup \{\infty\}$ denotes the one-point compactification of $\mathbb{A}_n = \mathbb{R}^{2^n}$.

Making the identification

$$\mathbb{A}_n \cup \{\infty\} = S^{2^n} = S(\mathbb{A}_n \times \mathbb{A}_0)$$

where the line at infinity is $\{(0, re_0) \in \mathbb{A}_n \times \mathbb{A}_0\}$ so $\hat{p}$ is a continuous map from $S(\mathbb{A}_n \times \mathbb{A}_0)$ to $S(\mathbb{A}_n \times \mathbb{A}_0)$.

Claim: $\hat{p}$ and ρ_k the k–power map, are homotopic.

Let us define

$$\begin{aligned} F_t(x) &= x^k(e_0 + (1-t)g(x)) \\ F_t(\infty) &= \infty \end{aligned}$$

for $0 \leq t \leq 1$. Obviously F_t is continuous on x and t and

$$F_0(x) = x^k(e_0 + g(x)) = p(x)$$

and $F_0(\infty) = \hat{p}(\infty) = \infty$ and $F_1(x) = x^k$ and $F_1(\infty) = \infty$.

Thus $\hat{p}$ and ρ_k are homotopic.

By corollary 2.5, ρ_k has degree k then $\hat{p}$ has degree k and $\hat{p}$ and p are onto, so for $0 \in \mathbb{A}_n$ there is α in $\mathbb{A}_n$ such that $p(\alpha) = 0$.

Q.E.D.

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